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AUTOMORPHISMS OF K3 SURFACES IN POSITIVE CHARACTERISTIC

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The main topic of this talk consists of extending the known results about automorphisms of complex algebraic K3 surfaces to the case when the ground field is an algebraically closed field of positive characteristic. We review first what is known in the case when the ground field is \mathbf{C} .

1. AUTOMORPHISMS OF COMPLEX PROJECTIVE K3 SURFACES

Recall that an algebraic K3 surface over an algebraically closed field k is a smooth projective surface X over k such that the canonical class K_X is zero and the first Betti number is zero. When $k = \mathbf{C}$ all K3 surfaces are diffeomorphic. The fundamental group is trivial and the second cohomology group $H^2(X, \mathbf{Z})$ is a free group of rank 22 equipped with an integral symmetric bilinear form defined by the cup product. The lattice $H^2(X, \mathbf{Z})$ is even, unimodular and of signature $(3, 19)$, hence isomorphic to $U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$, where $E_8(-1)$ is the lattice defined by the negative of Cartan matrix of simple root system of type E_8 and U the standard hyperbolic plane. The subgroup S_X of $H^2(X, \mathbf{Z})$ generated by algebraic 2-cycles is isomorphic to the Picard group $\text{Pic}(X)$ of line bundles on X . As a sublattice of $H^2(X, \mathbf{Z})$ it is an even, not necessarily unimodular lattice of signature $(1, r-1)$. It is called the Picard lattice, or Neron-Severi lattice of X . The Picard lattice is an important invariant of a K3 surface. The deficiency from the unimodularity is measured by the discriminant group $D(S_X)$ which is a finite abelian group equipped with a quadratic form with values in $\mathbf{Q}/2\mathbf{Z}$. It is more or less known which lattices can occur as the Picard lattice of a K3 surface.

Let $\text{Aut}(X)$ be the group of regular automorphisms of X (or, equivalently, holomorphic automorphisms, or birational automorphisms). The main tool in describing the structure of $\text{Aut}(X)$ is the celebrated Global Torelli Theorem due to I. Piatetsky-Shapiro and I. Shafarevich [PS]. According to this theorem, an isometry $\sigma : H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$ between the cohomology lattices of two K3 surfaces X and X' is induced by an isomorphism $f : X' \rightarrow X$, i.e. $\sigma = f^*$, if and only if σ sends an ample divisor of X to an ample divisor of X' and the linear extension of σ over \mathbf{C} sends the 1-dimensional space of holomorphic 2-forms $\Omega^2(X) \subset H^2(X, \mathbf{C})$ on X to $\Omega^2(X')$ on X' .

The group $\text{Aut}(X)$ has a natural representation in the orthogonal groups $O(H^2(X, \mathbf{Z}))$ and $O(S_X)$. The cohomology class $[C]$ of a smooth rational curve C on X is a vector in S_X of norm $([C], [C]) = -2$. This follows from the adjunction formula because $K_X = 0$. This implies that $[C]$ is a (-2) -root. Let W_X be the subgroup of $O(S_X)$

generated by reflections in the classes of smooth rational curves (or, equivalently, all (-2) -roots). It is called (-2) -reflection subgroup of $O(S_X)$.

Theorem 1.1. *Let $\rho : \text{Aut}(X) \rightarrow O(S_X)$, $f \mapsto f^*$, be the representation of $\text{Aut}(X)$ on the lattice of algebraic cycles. Let $\text{Aut}(X)^*$ be the image of ρ . Then $\text{Aut}(X)^* \cap W_X = 1$ and $G = \text{Aut}(X)^* \cdot W_X$ is of finite index in $O(S_X)$ and is equal to the semi direct product of $\text{Aut}(X)^*$ and W_X .*

The kernel of the representation ρ is contained in a group of projective automorphisms of X with respect to any projective embedding of X . It is known that X has no holomorphic vector fields, so this group must be finite. Thus Theorem 1 gives a criterion for finiteness of $\text{Aut}(X)$ in terms of the lattice S_X : the group $\text{Aut}(X)$ is finite if and only if the subgroup W_X is of finite index in $O(S_X)$. All abstract even lattices M of signature $(1, r-1)$ with the property that the subgroup generated by reflections in (-2) -roots is of finite index in $O(M)$ have been classified by V. Nikulin and E. Vinberg. All of them can be realized as the Picard lattice of some K3 surface. We call them 2-reflexive lattices. This gives, in principle, a classification of all K3 surfaces with finite automorphism group.

Let us describe more precise results about automorphisms of complex projective K3 surfaces. The automorphism group $\text{Aut}(X)$ acts on the 1-dimensional space of holomorphic 2-forms $\Omega^2(X)$ on X . Let $\chi : \text{Aut}(X) \rightarrow \mathbf{C}^*$ be the corresponding character. The image of χ is a finite cyclic group. (It may be infinite if X is non-projective.) All possible such finite cyclic groups were described by S. Kondo [Ko1] and S. Vorontsov [V].

Theorem 1.2. *The possible order of $\chi(\text{Aut}(X))$ is a divisor of one of the numbers 66, 44, 42, 36, 28, or 12, or a power of prime $p^k \leq 66$, where $p \leq 19$ and $k \leq 4$.*

An automorphism g is called symplectic if $\chi(g) = 1$. The name is explained by the fact that a non-zero holomorphic 2-form on X defines a symplectic structure on X , so a symplectic automorphism preserves a symplectic structure. The possible order n of a symplectic automorphism and the number f of its fixed points is given by V. Nikulin [N].

Theorem 1.3. *The possible pairs (n, f) are as follows.*

$$(n, f) = (2, 8), (3, 6), (4, 4), (5, 4), (6, 2), (7, 3), (8, 2).$$

More generally, a subgroup $G \subset \text{Aut}(X)$ is called symplectic if $G \subset \text{Ker}(\chi)$. The classification of abelian groups of symplectic automorphisms was first given by Nikulin. It was later extended to not necessarily abelian groups by S. Mukai [M].

Theorem 1.4. *Let G be a symplectic subgroup of $\text{Aut}(X)$ which is not contained in any other symplectic subgroup. Then G is isomorphic to one of the 11 groups*

$$PSL(2, \mathbf{F}_7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}.$$

Each case is supported by an example.

It is an amazing discovery of S. Mukai that each of the 11 groups can be realized as a subgroup of a sporadic simple group, the Mathieu group M_{23} . Among all subgroups of M_{23} these subgroups can be characterized by the property that they have ≥ 5 orbits in the natural action of M_{23} on the projective line $\mathbf{P}^1(\mathbf{F}_{23})$. Recall

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that M_{23} is realized as a stabilizer of a point in the Mathieu group M_{24} which is defined as a certain group acting quintuply transitively on the set $\mathbf{P}^1(\mathbf{F}_{23})$.

Later on, another proof of Theorem 1.4 was given by S. Kondo who found another remarkable connections with rank 24 unimodular negative definite lattices called Niemeier lattices [Ko2].

2. AUTOMORPHISMS OF K3 SURFACES IN POSITIVE CHARACTERISTIC

Now let X be a K3 surface over an algebraically closed field k of characteristic $p > 0$. The Picard lattice S_X (= Neron-Severi lattice) of X is defined to be the group of all divisors modulo numerical equivalence, which with the intersection pairing becomes a lattice. The main difficulty in extending the previous results to the case of positive characteristic is the absence of the Global Torelli Theorem. However, Theorem 1.1 admits a generalization to the case of supersingular K3 surfaces (A. Ogus [O]). Recall that in the complex case, Hodge theory gives that $r = \text{rank Pic}(X) \leq h^{1,1}(X) = 20$. The absence of the Hodge decomposition gives only $r \leq 22$. In the extreme case $r = 22$ the surface X is called supersingular. Also note that $r = 21$ does not occur.

In any characteristic, it is known that $\text{Aut}(X)^* \cap W_X = 1$ but it is not known that the semi direct product $\text{Aut}(X)^* \cdot W_X$ is of finite index in $O(S_X)$. Still we have

Theorem 2.1. *If $S_X = \text{Pic}(X)$ is 2-reflexive, i.e. W_X is of finite index in $O(S_X)$, then $\text{Aut}(X)$ is finite.*

The automorphism group $\text{Aut}(X)$ acts on the 1-dimensional space of regular 2-forms $\Omega^2(X)$ on X . Let $\chi : \text{Aut}(X) \rightarrow k^*$ be the corresponding character. Little is known about the image of χ .

An automorphism g is called symplectic if $\chi(g) = 1$. Next we consider symplectic automorphisms of X of finite order. If the order n is coprime to the characteristic p , then we have the same result as in the complex case.

Theorem 2.2. *Let g be a symplectic automorphism of finite order n . If $(n, p) = 1$, then g has only finitely many fixed points and the possible pairs (n, f) are the same as in Theorem 1.3, that is,*

$$(n, f) = (2, 8), (3, 6), (4, 4), (5, 4), (6, 2), (7, 3), (8, 2).$$

Proof. At a fixed point of g , g is linearizable because $(n, p) = 1$. This implies that the quotient surface $Y = X/g$ has at worst A_m -singularities and its minimal resolution is a K3 surface. So, Nikulin's argument [N] for the complex K3 surfaces works except the following 2 cases.

Case 1: $n = 11$ and X/g has 2 A_{10} -singularities.

Case 2: $n = 15$ and X/g has 3 singularities of type A_{14} , A_4 and A_2 .

(When the ground field is \mathbf{C} , both cases are ruled out easily by the fact that the Picard number of K3 surfaces cannot exceed 20.)

In both cases, let \tilde{Y} be the minimal resolution of $Y = X/g$. Then \tilde{Y} is a supersingular K3 surface. The Picard lattice $S_{\tilde{Y}}$ contains the sublattice of rank 20 coming from the resolution of singularities, $A_{10} \oplus A_{10}$ in Case 1 and $A_{14} \oplus A_4 \oplus A_2$ in Case 2. The discriminant group of the sublattice of rank 20 is an abelian group of length 2, non- p -elementary. On the other hand, the discriminant group of $S_{\tilde{Y}}$

is a p -elementary group of length 2σ , where σ is the Artin invariant of \tilde{Y} . This implies that $\sigma = 1$ and hence $S_{\tilde{Y}}$ is a p -elementary even lattice of rank 22, signature $(1, 21)$ and length 2. Now some calculation shows that such a lattice cannot contain $A_{10} \oplus A_{10}$ unless $p = 11$, and $A_{14} \oplus A_4 \oplus A_2$ unless $p = 3$ or $p = 5$. \square

The situation is completely different in the case when the order $n = p^k$. In this case, g is automatically symplectic, and is called wild. We have the following result (I. Dolgachev and J.Keum [DK]).

Theorem 2.3. *Let g be an automorphism of order p and let X^g be the set of fixed points of g . Then one of the following cases occur :*

- (1) X^g is finite and consists of 0, 1 or 2 points.
- (2) X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 0. In this case X^g is a nodal cycle, i.e. the union of smooth rational curves.
- (3) X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 1. In this case $p \leq 11$ and there exists a divisor D with support on X^g such that the linear system $|D|$ defines an elliptic or quasi-elliptic fibration $\phi : X \rightarrow \mathbf{P}^1$.
- (4) X^g is a divisor such that the Kodaira dimension of the pair (X, X^g) is equal to 2. In this case X^g is equal to the support of some nef and big divisor D . Take D minimal with this property. Let $d = \dim H^0(X, \mathcal{O}_X(D - X^g))$ and $N = \frac{1}{2}D^2 + 1$. Then $p(N - d - 1) \leq 2N - 2$.

In the complex case, the quotient of the surface by a finite group of symplectic automorphisms has only rational double points and is birationally isomorphic to a K3 surface. This is no longer true in positive characteristic [DK].

Theorem 2.4. *Let g be as in Theorem 2.3. If $|X^g| = 2$, $X/(g)$ is birationally a K3 surface. If $|X^g| = 1$, $X/(g)$ is birationally either a K3 surface or a rational surface. The K3 case can occur only if $p \leq 5$. If $\dim X^g > 0$, the quotient $X/(g)$ is always a rational surface.*

The following is the main result of this talk, which says that there is no wild p -cyclic action on a K3 surface if $p > 11$.

Theorem 2.5. *If X admits an automorphism g of order p , then $p \leq 11$.*

Proof. We give a sketch of proof. According to Theorem 2.3-2.4 and some detailed analysis in [Dolgachev-Keum], we are reduced to prove the bound in the following two cases :

Case 1 : $|X^g| = 1$ and $X/(g)$ is a rational surface with an elliptic Gorenstein singularity.

Case 2 : X^g is a nodal cycle and $X'/(g)$ is a rational surface with an elliptic Gorenstein singularity, where X' is the orbifold K3 surface obtained by contracting X^g to a rational double point.

We claim that in both cases X admits a g -stable elliptic fibration such that X^g is contained in a fibre.

For simplicity, let's consider the first case. The second case can be handled similarly.

Note that the quotient surface $Y = X/(g)$ has trivial canonical divisor. Let

$$\sigma : \tilde{Y} \rightarrow Y$$

be the minimal resolution. Then

$$K_{\tilde{Y}} = -\Delta,$$

where Δ is an effective divisor whose reduced divisor is the exceptional set of σ . There is a birational morphism

$$\phi : \tilde{Y} \rightarrow Z$$

onto a rational surface Z with $K_Z^2 = 0$. More precisely, ϕ is a blow up of points on simple components of a member F of the anti-canonical system $|-K_Z|$. Then

$$\phi^*F = \Delta + \sum a_i E_i$$

for some positive integers a_i , where $\cup E_i$ is the exceptional set of ϕ . Note that the intersection number

$$(\phi^*F)\Delta_i = 0$$

for every component Δ_i of Δ . Denote by $\overline{E_i}$ the image in Y of E_i , and consider the effective Weil divisor $D := \sum a_i \overline{E_i}$ on Y . Calculating Mumford's intersection number on a normal surface, we have

$$\sigma^*D = \phi^*F,$$

and hence $D^2 := (\sigma^*D)^2 = F^2 = 0$. Since Y is \mathbb{Q} -factorial, the pull back π^*D , where $\pi : X \rightarrow Y = X/g$ is the quotient morphism, is an effective integral divisor of self intersection 0. Let A be an effective integral divisor of self intersection 0, proportional to π^*D and not divisible in $\text{Pic}(X)$. Then A is the sum of a fibre C of an elliptic fibration and possibly, sections, S_1, S_2, \dots, S_t . Here we assume $p > 11$. Let $A = P + N$ be the Zariski decomposition, i.e., P is numerically effective, $PN_i = 0$ for each component of N , and the intersection matrix $(N_i N_j)$ is negative definite. Since $g^*(\pi^*D) = \pi^*D$ (equal as divisors), $g^*A = A$ as divisors. The uniqueness of the Zariski decomposition implies that $g^*P = P$ as divisors. Note that in our case $P = C + \sum S_i/2$. So $g^*C = C$, i.e., the elliptic pencil $|C|$ is g -stable. Now it is easy to see that the fibre C contains X^g . This proves the claim.

A similar argument as in [Section 5, DK] shows that X admits an automorphism g of order p , the characteristic, and a g -stable elliptic fibration only if $p \leq 11$. This contradicts to the assumption $p > 11$. \square

The main observation of Mukai in proving Theorem 1.4 is that the number of fixed points of a symplectic automorphism of order n is equal to

$$\epsilon(n) = 24(n \prod_{p|n} (1 + \frac{1}{p}))^{-1}.$$

A Mathieu representation of a group G is a 24-dimensional representation with character $\chi(g) = \epsilon(\text{ord}(g))$. The natural action of a finite group G of symplectic automorphisms of a complex K3 surface on $H^*(X, \mathbb{Q}) \cong \mathbb{Q}^{24}$ is a Mathieu representation. From this Mukai deduces that G is isomorphic to a subgroup of M_{23} with at least 5 orbits. In positive characteristic the formula for the number of fixed points is no longer true and the representation of G on the l -adic cohomology $H_{\text{et}}^*(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ is not Mathieu in general for any $l \neq p$. For $p > 11$, from Theorem 2.2 and 2.5, we have the following :

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Theorem 2.6. *Let G be a finite group of symplectic automorphisms of a K3 surface in characteristic $p > 11$. Then the natural representation of G on $H_{\text{et}}^*(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ is Mathieu for $l \neq p$.*

Even in the case of characteristic $p > 11$, one cannot expect the same classification of finite symplectic groups G as in Mukai's list. The problem is the existence of supersingular K3 surfaces. For such surfaces, the G -invariant subspace of $H_{\text{et}}^*(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ has dimension ≥ 3 . Nevertheless, we hope to have a complete classification of finite symplectic groups in positive characteristic.

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